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On the Degeneracy of Topological Index *J*

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On the Degeneracy of Topological Index J[#]

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Abstract

The topological index J is one of the widely used indices for QSAR and QSPR studies, and has the advantage of discriminating nonisomorphic graphs that represent molecules of restricted size. It is of interest to see if this property can be generalized for any connected graph. A proof is provided for the theorem stating that for almost all connected graphs G there is a nonisomorphic graph G' such that G and G' have the same topological index J. Since the index J is among the least degenerate topological indices, our results imply that *a fortiori* other topological indices are even more degenerate. Hence, none of them can be used in discrimination of general graphs.

Keywords. Balaban index; topological index J; degeneracy; topological index; molecular descriptor.

1 INTRODUCTION

Among molecular descriptors, topological indices [1–9] occupy a well–established place and are frequently used for quantitative structure–activity relationship (QSAR) studies [10]. By analogy, one uses similar acronyms for physical–chemical properties (QSPR) or toxicities (QSTR).

For saturated hydrocarbons, their connected and hydrogen–depleted graph consists in vertices symbolizing carbon atoms and edges symbolizing single C–C covalent bonds. The topological index J [11–12] (average distance sum connectivity) was modeled after the Randić index χ [13] and it was designed to be both less degenerate (by using topological distance sums instead of vertex degrees as for χ), and less dependent on the graph size and cyclicity (by compensating through the use of the number of edges and the cyclomatic number μ).

Indeed, for an infinitely long linear graph (idealized polyethylene) it was proved [14] that the value of J becomes equal to the number π, and for idealized infinite polymers having chains with

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regularly placed branches such as poly(propylene) or poly(isobutene), the value of *J* is a rational multiple of π (in the two above examples *J* is 3π/2and 2π, respectively).

A computer program was published for computing index J [15]. For molecules having heteroatoms and/or multiple bonds, suitable changes were proposed [16–18]. It was found that the index *J* parallels in many ways the less discriminating Wiener index, *e.g*. in the way these indices order alkane isomers [19,20]. An application of *J* for polymers was described [21]. The index J has found many applications; the most interesting of these was developed by Lahana and his coworkers [22], and it involved index *J* along with 12 other molecular descriptors for reducing the number of possible immunosuppressive decapeptides in a virtual library of about 250,000, by a factor of 10,000, resulting in improving the activity of the lead decapeptide about 100 times. Another interesting application of *J* and χ was published by Bermudez *et al*. [23], and it involved adding to these indices contributions of hydrogen bonding for computing pairwise associations between nucleotide bases in transfer RNA of *E. coli* for clustering tRNA into two groups, confirming Wong's coevolution theory of the genetic code [24]: biological evolution led to gradually increasing numbers of amino acids till they reached the present number of 20, and thus the 64 triplet codons had to be reassigned from time to time according to evolving biochemical pathways.

In a previous publication [25], it was proved that the highly discriminating index *J* starts to become degenerate for acyclic graphs with $n > 11$ (n is the number of vertices), for monocyclic graphs with $n > 7$, and for polycyclic graphs with even smaller *n* values. Thus, despite the fact that *J* has a higher discriminating ability than most other single topological indices, it may be inferred that for almost any graph *J* is degenerate. A formal proof of this conjecture is the object of the present paper.

2 NOTATIONS

Let x be any real number. By $\lfloor x \rfloor$ we denote the largest integer smaller than x and by $\lceil x \rceil$ we denote the smallest integer larger than x. Let S be any finite set; by $|S|$, we denote its cardinality, i. e. the number of elements in S. Let G be a simple connected graph (i. e. a connected graph without loops or multiple edges). By $V(G)$ we denote the set of vertices of G, and by $E(G)$ the set of edges of G. Also we denote their cardinalities by $v(G)=|V(G)|$ and $e(G)=|E(G)|$. Let $x, y \in V(G)$. By $d_G(x)$, we denote the degree of vertex x, and by $d_G(x, y)$ the topological distance between vertices x and y. We also define the sum of all distances from vertex x by $D_G(x)=\sum_{v\in V(G)} d_G(x,v)$.

The Balaban index $J(G)$ is defined by

$$J(G)=\frac{e(G)}{e(G)-(v(G)-1)+1}\cdot\sum_{uv\in E(G)}\left(D_G(u)D_G(v)\right)^{-1/2} \tag{1}$$

For the sake of simplicity, we define $E_G(uv) = (D_G(u) D_G(v))^{-1/2}$. The last relationship can be rewritten as

$$J(G) = \frac{e(G)}{e(G) - (v(G) - 1) + 1} \cdot \sum_{e \in E(G)} E_G(e) \tag{2}$$

Let G_n be the set of all simple graphs G such that $V(G) = SV = \{v_1, v_2, ..., v_n\}$, where n is an arbitrary number. Note that $n = v(G)$ for each graph in the class G_n. Denote by:

1) C_n the set of all connected graphs with the set of vertices SV.

2) D_n the set of all graphs with the set of vertices SV such that at least two of them are not connected and have no common neighbors.

3) D_n' the set of all graphs with the set of vertices SV such that v_1 and v_2 are not connected and have no common neighbors.

4) L_n the set of all graphs with the set of vertices SV with maximum degree at least $\frac{1}{2}n + n^{2/3}$.

5) L'_n the set of all graphs with the set of vertices SV such that $d_G(v_1) \geq \frac{1}{2}n + n^{2/3}$.

6) S_n the set of all graphs with the set of vertices SV with minimum degree at most $\frac{1}{2}n - n^{2/3}$.

7) S'_n the set of all graphs with the set of vertices SV such that $d_G(v_1) \leq \frac{1}{2}n - n^{2/3}$.

8) W_n the set of connected graphs G with the set of vertices SV such that there is a graph G' such that G and G' are nonisomorphic and that they have the same Balaban index.

More formally, we can write:

$$\begin{aligned}
C_n &= \{G \in G_n : G \text{ is connected graph}\} \\
D_n &= \{G \in G_n : G \notin C_n \text{ or } diam(G) > 2\} \\
D'_n &= \{G \in G_n : \text{vertices } \mathrm{v}_1 \text{ and } \mathrm{v}_2 \text{ are not connected or } d_G(v_1, v_2) > 2\} \\
L_n &= \left\{G \in G_n : \text{ the maximum degree of } G \text{ is at least } \frac{1}{2}n + n^{2/3}\right\} \\
L'_n &= \left\{G \in G_n : d_G(v_1) \geq \frac{1}{2}n + n^{2/3}\right\} \\
S_n &= \left\{G \in G_n : \text{the minimum degree of } G \text{ is at most } \frac{1}{2}n - n^{2/3}\right\} \\
S'_n &= \left\{G \in G_n : d_G(v_1) \leq \frac{1}{2}n - n^{2/3}\right\} \\
W_n &= \left\{\begin{array}{l} G \in C_n : \text{ there is a graph } G' \text{ such that } G \text{ and } G' \text{ are} \\ \qquad\qquad \text{nonisomorphic and } J(G) = J(G') \end{array}\right\}
\end{aligned} \tag{3}$$

In order to prove our main result, we shall need the Stirling formula:

$$\lim_{n\to\infty}\frac{n!}{\left(\frac{n}{e}\right)^n\sqrt{2\pi n}}=1 \tag{4}$$

3 RESULTS AND DISCUSSION

Let us give an auxiliary result:

Lemma Let a and b be arbitrary natural numbers and let $x_1, x_2, ..., x_a$ be any fixed real numbers. The number of sums of b summands from $\{x_1, ..., x_a\}$ is at most $\min\left\{a^b, (b+1)^a\right\}$.

Proof: The number of different sums is at most the number of ordered b–tuples from the set $\{x_1, ..., x_a\}$, hence it is at most a^b. Note that each sum can be rewritten in the form $k_1 \cdot x_1 + k_2 \cdot x_2 + ... + k_a \cdot x_a$, where k_i is the number of occurrences of the number x_i in the observed sum, for each $i = 1, ..., a$. Also, note that $0 \le k_i \le b$, for each $i = 1, ..., a$. Hence each of the sums is uniquely determined by the a–tuple of the numbers $(k_1, k_2, ..., k_a)$ where $0 \le k_i \le b$, for each $i = 1, ..., a$ and $k_1 + ... + k_a = b$. The number of these a–tuples is at most $(b+1)^a$. This lemma is proved.

Let us prove the following theorem:

Theorem 1. We have $\lim_{n\to\infty}\frac{|W_n|}{|C_n|}=1$.

Proof: The last relation is equivalent to $\lim_{n\to\infty}\frac{|C_n \setminus W_n|}{|C_n|}=0$. Obviously, $\lim_{n\to\infty}\frac{|C_n \setminus W_n|}{|C_n|}\ge 0$. Hence it is sufficient to prove that

$$\lim_{n\to\infty}\frac{|C_n \setminus W_n|}{|C_n|}\le 0 \tag{5}$$

Note that $|C_n \setminus W_n| \le |G_n \setminus W_n| \le |D_n| + |L_n| + |S_n| + |(G_n \setminus W_n)\setminus(D_n \cup L_n \cup S_n)|$, hence

$$\begin{aligned}
&\lim_{n\to\infty}\frac{|C_n \setminus W_n|}{|C_n|}=\lim_{n\to\infty}\frac{|C_n \setminus W_n|}{|G_n|}\cdot\frac{|G_n|}{|C_n|}\le \\
&\le \lim_{n\to\infty}\frac{|D_n| + |L_n| + |S_n| + |(G_n \setminus W_n)\setminus(D_n \cup L_n \cup S_n)|}{|G_n|}\cdot\frac{|G_n|}{|G_n|-|G_n \setminus C_n|}\le \\
&\le \lim_{n\to\infty}\frac{|D_n| + |L_n| + |S_n| + |(G_n \setminus W_n)\setminus(D_n \cup L_n \cup S_n)|}{|G_n|}\cdot\frac{|G_n|}{|G_n|-|D_n|}
\end{aligned}$$

Therefore, it is sufficient to prove that

$$\lim_{n\to\infty}\frac{|D_n|}{|G_n|}\le 0 \tag{6}$$

$$\lim_{n\to\infty}\frac{|L_n|}{|G_n|}\le 0 \tag{7}$$

$$\lim_{n\to\infty}\frac{|S_n|}{|G_n|}\le 0 \tag{8}$$

$$\lim_{n\to\infty}\frac{|(G_n\setminus W_n)\setminus(D_n\cup L_n\cup S_n)|}{|G_n|}\le 0 \tag{9}$$

Let us prove relationship (6). We have

$$\lim_{n\to\infty}\frac{|D_n|}{|G_n|}\le\lim_{n\to\infty}\frac{\binom{n}{2}\cdot|D'_n|}{|G_n|}$$

Let G be any graph in D'_n. Note that $v_1v_2\notin E(G)$. Also for each $v_j, j=3,...,n$ at least one of the edges v_1v_j and v_2v_j is not in $E(G)$. Therefore $|D'_n|=3^{n-2}\cdot 2^{\frac{(n-2)(n-3)}{2}}$. It follows that

$$\lim_{n\to\infty}\frac{\binom{n}{2}\cdot|D'_n|}{|G_n|}=\lim_{n\to\infty}\frac{\binom{n}{2}\cdot 3^{n-2}\cdot 2^{\frac{(n-2)(n-3)}{2}}}{2^{\frac{n(n-1)}{2}}}=\lim_{n\to\infty}\frac{\binom{n}{2}\cdot 3^{n-2}}{2^{2n-3}}=$$

$$=\lim_{n\to\infty}\left[\binom{n}{2}\cdot\frac{8}{9}\cdot\left(\frac{3}{4}\right)^n\right]=\lim_{n\to\infty}\left[\sqrt[n]{\binom{n}{2}\cdot\frac{8}{9}}\cdot\left(\frac{3}{4}\right)\right]^n$$

Note that $\lim_{n\to\infty}\left[\sqrt[n]{\binom{n}{2}\cdot\frac{8}{9}}\cdot\left(\frac{3}{4}\right)\right]=\frac{3}{4}$. Therefore, $\lim_{n\to\infty}\left[\sqrt[n]{\binom{n}{2}\cdot\frac{8}{9}}\cdot\left(\frac{3}{4}\right)\right]^n=0$.

Now, let us prove relationship (7). We have

$$\lim_{n\to\infty}\frac{|L_n|}{|G_n|}\le\lim_{n\to\infty}\frac{n\cdot|L'_n|}{|G_n|}$$

Let G be any graph in L'_n. Note that at least $\frac{1}{2}n+n^{2/3}$ of edges in $\{v_1v_2, v_1v_3,...,v_1v_n\}$ are in $E(G)$. Hence, we have less than $\binom{n-1}{\left\lceil\frac{1}{2}n+n^{2/3}\right\rceil}$ choices for these edges. Therefore,

$$\lim_{n\to\infty}\frac{|L_n|}{|G_n|}\le\lim_{n\to\infty}\frac{n\cdot\binom{n-1}{\left\lceil\frac{1}{2}n+n^{2/3}\right\rceil}\cdot 2^{\frac{(n-1)(n-2)}{2}}}{2^{\frac{n(n-1)}{2}}}=\lim_{n\to\infty}\frac{n\cdot\binom{n-1}{\left\lceil\frac{1}{2}n+n^{2/3}\right\rceil}}{2^{n-1}}\le\lim_{n\to\infty}\frac{n\cdot\binom{n}{\left\lceil\frac{1}{2}n+n^{2/3}\right\rceil}}{2^{n-1}}$$

$$=\lim_{n\to\infty}\frac{n}{2^{n-1}}\cdot\frac{n!}{\left(\left\lceil\frac{1}{2}n+n^{2/3}\right\rceil\right)!\cdot\left(\left\lfloor\frac{1}{2}n-n^{2/3}\right\rfloor\right)!}$$

$$= \lim_{n\to\infty} \frac{n\cdot\left(\frac{n}{e}\right)^{n}\cdot\sqrt{2\pi n}}{2^{n-1}\cdot\left[\left(\frac{\left\lceil \frac{1}{2}n+n^{2/3}\right\rceil}{e}\right)^{\left\lceil \frac{1}{2}n+n^{2/3}\right\rceil}\cdot\sqrt{2\pi\left\lceil \frac{1}{2}n+n^{2/3}\right\rceil}\cdot\left(\frac{\left\lfloor \frac{1}{2}n-n^{2/3}\right\rfloor}{e}\right)^{\left\lfloor \frac{1}{2}n-n^{2/3}\right\rfloor}\cdot\sqrt{2\pi\left\lfloor \frac{1}{2}n-n^{2/3}\right\rfloor}\right]}$$

$$= \lim_{n\to\infty}\frac{n}{2^{n-1}}\cdot\frac{\sqrt{n}}{\sqrt{\left\lceil \frac{1}{2}n+n^{2/3}\right\rceil}\sqrt{2\pi\left\lfloor \frac{1}{2}n-n^{2/3}\right\rfloor}}\cdot\frac{n^{n}}{\left\lceil \frac{1}{2}n+n^{2/3}\right\rceil^{\left\lceil \frac{1}{2}n+n^{2/3}\right\rceil}\cdot\left\lfloor \frac{1}{2}n-n^{2/3}\right\rfloor^{\left\lfloor \frac{1}{2}n-n^{2/3}\right\rfloor}}$$

$$\le \lim_{n\to\infty}\frac{2n^{3/2}}{\sqrt{\left(\frac{1}{2}n+n^{2/3}\right)}\sqrt{2\pi\left(\frac{1}{2}n-n^{2/3}-1\right)}}\cdot\frac{1}{\left(1+2n^{-1/3}\right)^{\frac{1}{2}n+n^{2/3}}\cdot\left(1-2n^{-1/3}-n^{-1}\right)^{\frac{1}{2}n-n^{2/3}-1}}$$

$$= \lim_{n\to\infty}\frac{2n^{3/2}\cdot\left(1-2n^{-1/3}-n^{-1}\right)}{\sqrt{\left(\frac{1}{2}n+n^{2/3}\right)}\sqrt{2\pi\left(\frac{1}{2}n-n^{2/3}-1\right)}}\cdot\frac{1}{\left(1-4n^{-2/3}-n^{-1}-2n^{-4/3}\right)^{\frac{1}{2}n}\cdot\left(\frac{1+2n^{-1/3}}{1-2n^{-1/3}-n^{-1}}\right)^{n^{2/3}}}$$

$$\le \lim_{n\to\infty}\frac{2n^{3/2}}{\sqrt{\left(\frac{1}{2}n+n^{2/3}\right)}\sqrt{2\pi\left(\frac{1}{2}n-n^{2/3}-1\right)}}\cdot$$

$$\cdot\frac{1}{\left[\left(1-\frac{1}{\frac{1}{4}n^{2/3}}\right)^{\frac{1}{4}n^{2/3}\cdot\frac{\frac{1}{2}n}{\frac{1}{4}n^{2/3}}}\cdot\left(1+\frac{1}{\frac{1-2n^{-1/3}}{4n^{-1/3}}}\right)^{\frac{1-2n^{-1/3}}{4n^{-1/3}}\cdot\frac{n^{2/3}}{\frac{1-2n^{-1/3}}{4n^{-1/3}}}}\cdot\left(\frac{1-4n^{-2/3}-n^{-1}-2n^{-4/3}}{1-4n^{-2/3}}\right)^{\frac{1}{2}n}\cdot\left(\frac{1-2n^{-1/3}}{1-2n^{-1/3}-n^{-1}}\right)^{n^{2/3}}\right]}$$

$$= \lim_{n\to\infty}\frac{2n^{3/2}}{\sqrt{\left(\frac{1}{2}n+n^{2/3}\right)}\sqrt{2\pi\left(\frac{1}{2}n-n^{2/3}-1\right)}}\cdot\frac{1}{e^{-2n^{1/3}}\cdot e^{\frac{4n^{1/3}}{1-2n^{-1/3}}}}\cdot$$

$$\cdot\frac{1}{\left(1-\frac{n^{-1}+2n^{-4/3}}{1-4n^{-2/3}}\right)^{\frac{1}{\frac{n^{-1}+2n^{-4/3}}{1-4n^{-2/3}}}\cdot\frac{n^{-1}+2n^{-4/3}}{1-4n^{-2/3}}\cdot n}\cdot\frac{1}{\left(1-\frac{n^{-1}}{1-2n^{-1/3}}\right)^{\frac{1}{\frac{n^{-1}}{1-2n^{-1/3}}}\cdot\frac{n^{-1}}{1-2n^{-1/3}}\cdot n^{2/3}}}}$$

$$= \lim_{n\to\infty}\frac{2n^{3/2}}{\sqrt{\left(\frac{1}{2}n+n^{2/3}\right)}\sqrt{2\pi\left(\frac{1}{2}n-n^{2/3}-1\right)}}\cdot\frac{1}{e^{\frac{2n^{1/3}+4}{1-2n^{-1/3}}}}\cdot\frac{1}{e^{-1}}$$

$$= \lim_{n\to\infty}\left[\left(\frac{2n^{3/2}\cdot e}{\sqrt{\left(\frac{1}{2}n+n^{2/3}\right)}\sqrt{2\pi\left(\frac{1}{2}n-n^{2/3}-1\right)}}\right)^{\frac{1}{n^{1/3}}}\cdot\frac{1}{e^{\frac{2+4n^{-1/3}}{1-2n^{-1/3}}}}\right]^{n^{1/3}}=0$$

Now, let us prove relationship (8). We have

$$\lim_{n\to\infty}\frac{|S_n|}{|G_n|}\le\lim_{n\to\infty}\frac{n\cdot|S'_n|}{|G_n|}$$

Let G be any graph in S'_n. Note that at most $\frac{1}{2}n-n^{2/3}$ of edges in $\{v_1v_2, v_1v_3, ..., v_1v_n\}$ are in $E(G)$. Therefore, we have less than $\binom{n-1}{\frac{1}{2}n-n^{2/3}}$ choices for these edges. Therefore,

$$\lim_{n\to\infty}\frac{|L_n|}{|G_n|}\le\lim_{n\to\infty}\frac{n\cdot\binom{n-1}{\left\lfloor\frac{1}{2}n-n^{2/3}\right\rfloor}\cdot 2^{\frac{(n-1)(n-2)}{2}}}{2^{\frac{n(n-1)}{2}}}\le\lim_{n\to\infty}\frac{n\cdot\binom{n}{\left\lfloor\frac{1}{2}n-n^{2/3}\right\rfloor}}{2^{n-1}}=$$

$$\lim_{n\to\infty}\frac{n\cdot\binom{n}{\left\lceil\frac{1}{2}n+n^{2/3}\right\rceil}}{2^{n-1}}=\{\text{as it is earlier proved}\}=0.$$

It remains to prove relationship (9). Let us bound from above $\left|(G_n\setminus W_n)\setminus(D_n\cup L_n\cup S_n)\right|$. Denote $A_n = \{J(G) : G \in (G_n \setminus W_n) \setminus (D_n \cup L_n \cup S_n)\}$. Note that for each graph $G\in G_n$ there are at most $n!$ graphs G' such that G and G' are isomorphic. Therefore,

$$\left|(G_n\setminus W_n)\setminus(D_n\cup L_n\cup S_n)\right|\le n!\cdot|A_n|$$

Let G be an arbitrary graph from the set $(G_n\setminus W_n)\setminus(D_n\cup L_n\cup S_n)$. Note that each vertex $v\in V(G)$ has more than $\frac{1}{2}n-n^{2/3}$ and less than $\frac{1}{2}n+n^{2/3}$ neighbors and all other vertices from $V(G)\setminus\{v\}$ are on distance 2. Therefore,

$$\frac{1}{2}n+n^{2/3}+2\left(n-1-\left(\frac{1}{2}n+n^{2/3}\right)\right)<D_G(v)<\frac{1}{2}n-n^{2/3}+2\left(n-1-\left(\frac{1}{2}n-n^{2/3}\right)\right)$$

$$\frac{3}{2}n-2-n^{2/3}<D_G(v)<\frac{3}{2}n-2+n^{2/3}$$

Hence there are less than $\lfloor 2n^{2/3}\rfloor$ possible values of $D_G(v)$. Therefore for each $e\in E(G)$, there are less than $\lfloor 4n^{4/3}\rfloor$ possible values of $E_G(e)$.

Let us observe the formula $J(G)=\frac{e(G)}{e(G)-(v(G)-1)+1}\cdot\sum_{e\in E(G)}E_G(e)$. The value of the first factor

$\frac{e(G)}{e(G)-(v(G)-1)+1}$ can be chosen in $\frac{n(n-1)}{2}+1$ ways ($v(G)$ is prescribed and $0 \le e(G) \le \frac{n(n-1)}{2}$). The value of the second factor $\sum_{e\in E(G)} E_G(e)$ can be chosen in at most $\left(\frac{n(n-1)}{2}+1\right)^{\lfloor 4n^{4/3}\rfloor} \le \left(\frac{n(n-1)}{2}+1\right)^{4n^{4/3}}$ ways. (Note that there are at most $\frac{n(n-1)}{2}$ summands and each of them can take one of $\lfloor 4n^{4/3} \rfloor$ values and use Lemma 1). Therefore, $|A_n| \le \left(\frac{n\cdot(n-1)}{2}\right)\left(\frac{n\cdot(n-1)}{2}+1\right)^{4n^{4/3}} \le n^2 \cdot n^{8n^{4/3}} \le n^{10n^{4/3}}$.

Hence,

$$\left|(G_n \setminus W_n)\setminus(D_n \cup L_n \cup S_n)\right| \le n! \cdot n^{10n^{4/3}}$$

It is sufficient to prove that

$$\lim_{n\to\infty} \frac{n! \cdot n^{10n^{2/3}}}{2^{\frac{n(n-1)}{2}}} \le 0$$

Let us prove this

$$\lim_{n\to\infty} \frac{n! \cdot n^{10n^{4/3}}}{2^{\frac{n(n-1)}{2}}} \le \lim_{n\to\infty} \frac{n^n \cdot n^{10n^{4/3}}}{2^{\frac{n(n-1)}{2}}} \le \lim_{n\to\infty} \frac{n^{11n^{4/3}}}{2^{\frac{n(n-1)}{2}}} = \lim_{n\to\infty} \left(\frac{n^{11}}{2^{\frac{n(n-1)}{2n^{4/3}}}}\right)^{n^{4/3}}$$

Note that $\lim_{n\to\infty} \left(\frac{n^{11}}{2^{\frac{n(n-1)}{2n^{4/3}}}}\right) = 0,$ hence indeed $\lim_{n\to\infty} \left(\frac{n^{11}}{2^{\frac{n(n-1)}{2n^{4/3}}}}\right)^{n^{4/3}} = 0$. This proves our theorem.

We would like to establish what portion of connected graphs are graphs that have nonisomorphic graphs with the same Balaban index, more formally, we would like to calculate $\left|\bigcup_{n\in N} W_n\right| / \left|\bigcup_{n\in N} C_n\right|$. Unfortunately, this is impossible, because both nominator and denominator of the last ratio are infinitesimal. The best that we can do is to calculate

$$\lim_{x\to\infty} \left|\bigcup_{n=1}^{x} W_n\right| / \left|\bigcup_{n=1}^{x} C_n\right|$$

It is obvious that $\lim_{x\to\infty} \left|\bigcup_{n=1}^{x} W_n\right| / \left|\bigcup_{n=1}^{x} C_n\right| \le 1$. In order to prove that $\lim_{x\to\infty} \left|\bigcup_{n=1}^{x} W_n\right| / \left|\bigcup_{n=1}^{x} C_n\right| = 1$, it is sufficient to prove that for each $\varepsilon > 0$, we have $\lim_{x\to\infty} \left|\bigcup_{n=1}^{x} W_n\right| / \left|\bigcup_{n=1}^{x} C_n\right| > 1-\varepsilon$. Denote by $n_0 = n_0(\varepsilon)$ the smallest natural number such that we have $\frac{|W_n|}{|C_n|} > 1-\frac{\varepsilon}{2}$ for each $n > n_0$. It follows that:

$$\lim_{x\to\infty}\left|\bigcup_{n=1}^{x} W_n\right| / \left|\bigcup_{n=1}^{x} C_n\right| = \lim_{x\to\infty}\frac{\sum_{n=1}^{x}|W_n|}{\sum_{n=1}^{x}|C_n|} > \lim_{x\to\infty}\frac{\sum_{n=n_0+1}^{x}|C_n|\cdot\left(1-\frac{\varepsilon}{2}\right)}{\sum_{n=1}^{n_0}|C_n| + \sum_{n=n_0+1}^{x}|C_n|} =$$

$$= \lim_{x\to\infty}\frac{\frac{\varepsilon}{2}\cdot\sum_{n=n_0+1}^{x}|C_n| + \sum_{n=n_0+1}^{x}|C_n|\cdot(1-\varepsilon)}{\sum_{n=1}^{n_0}|C_n| + \sum_{n=n_0+1}^{x}|C_n|}$$

Note that $\sum_{n=1}^{n_0}|C_n|$ is a finite number and that $\lim_{x\to\infty}\sum_{n=n_0+1}^{x}|C_n|$ is an infinite number. Therefore, we can conclude that $\lim_{x\to\infty}\sum_{n=n_0+1}^{x}|C_n|\cdot\frac{\varepsilon}{2} > (1-\varepsilon)\sum_{n=1}^{n_0}|C_n|$. It follows that

$$\lim_{x\to\infty}\frac{\frac{\varepsilon}{2}\cdot\sum_{n=n_0+1}^{x}|C_n| + \sum_{n=n_0+1}^{x}|C_n|\cdot(1-\varepsilon)}{\sum_{n=1}^{n_0}|C_n| + \sum_{n=n_0+1}^{x}|C_n|} > \lim_{x\to\infty}\frac{(1-\varepsilon)\sum_{n=1}^{n_0}|C_n| + \sum_{n=n_0+1}^{x}|C_n|\cdot(1-\varepsilon)}{\sum_{n=1}^{n_0}|C_n| + \sum_{n=n_0+1}^{x}|C_n|} = 1-\varepsilon$$

Hence, indeed $\lim_{x\to\infty}\left|\bigcup_{n=1}^{x} W_n\right| / \left|\bigcup_{n=1}^{x} C_n\right| = 1$.

4 CONCLUSIONS

From the discussion above, the following conclusion follows: *For almost all connected graphs G there is a nonisomorphic graph G' such that $J(G) = J(G')$*. Since index *J* is among the least degenerate topological indices, our results imply that *a fortiori* other topological indices are even more degenerate for general graphs. It remains to be seen whether indices with even lower degeneracy than *J*, such as BCUT [26–28], obey the theorem proved here.

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